

Hamiltonian anomalies of bound states in QED.

V.I. Shilin^{1,2} and V.N. Pervushin¹

¹*Joint Institute for Nuclear Research, Dubna, Russia*

²*Moscow Institute of Physics and Technology, Dolgoprudny, Russia*

The Bound State in QED is described in systematic way by means of nonlocal irreducible representations of the nonhomogeneous Poincare group and Dirac's method of quantization. As an example of application of this method we calculate triangle diagram $Para - Positronium \rightarrow \gamma\gamma$. We show that the Hamiltonian approach to Bound State in QED leads to anomaly-type contribution to creation of pair of parapositronium by two photon.

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1. INTRODUCTION

The bound states in gauge theories are usually considered in the framework of representations of the homogeneous Lorentz group in one of the Lorentz-invariant gauges [1]. In this paper, we suggest a systematic scheme of the bound state generalization of S-matrix elements, which is based on irreducible representations of the nonhomogeneous Poincaré group in concordance with the first Quantum Electrodynamics (QED) quantization [2, 3] and first QED description of bound states [4, 5]. We obtain bound states by means of excluding time-component of four-potential [2, 3, 6] and Hubbard-Stratonovich transformation [7, 8], that is in agreement with general principles [9]. The time component is chosen in correspondence with the Markov-Yukawa constraint of irreducibility [10, 11].

The aim of the article is to research the experimental consequences of such the Poincaré group irreducible representations of QED (see also review [12]).

As a test of this scheme we calculate process $P \rightarrow \gamma\gamma$, where P is parapositronium, that describes triangle diagram, with one real and the other virtual photons.

The structure of the article is as follows. In section 2 we exclude time-component of four-potential from QED action. For derived action in section 3 we make the Hubbard-Stratonovich transformation and positronium field appear. Than we make semi-classical

quantization of the resulting system in section 4. In section 5 we calculate the triangle anomaly. And in section 6 we calculate the contribution in process $\gamma\gamma \rightarrow PP$ inspired by triangle anomaly.

2. EXCLUDING TIME COMPONENT OF A_μ IN QED

We start with usual Quantum Electrodynamics Lagrangian:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu - \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \quad (1)$$

Below we will work in frame of reference in which the bound state as whole is at rest.

The Lagrangian (1) contains nonphysical degree of freedom. This degree can be excluded by substitution the solution of classical equation of motion in action.

Variation action providing by (1) over A_0 leads to one component of Maxwell equation:

$$\Delta A_0 - \partial_0\partial_k A_k = -j_0,$$

where $\Delta = \partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3$. If we take the gauge:

$$\partial_k A_k = 0 \quad (2)$$

than the previous equation simplifies:

$$\Delta A_0 = -j_0.$$

The solution of this equation is:

$$A_0 = -\frac{1}{\Delta}j_0 \quad (3)$$

where

$$\left(\frac{1}{\Delta}j\right)(x, y, z) = -\frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus \{(x, y, z)\}} dx' dy' dz' \frac{j(x', y', z')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

Under gauge fixing condition (2) Lagrangian (1) has the form:

$$\mathcal{L} = \frac{1}{2}\partial_\mu A_m \partial^\mu A_m - \frac{1}{2}A_0 \Delta A_0 - A_0 j_0 + A_m j_m - \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

Substitute into this Lagrangian the solution of the classical equation of motion (3):

$$S = \int d^4x \left(\frac{1}{2}(\dot{A}_i \dot{A}_i - B_i B_i) + A_i j_i - \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}j_0 \frac{1}{\Delta}j_0 \right) \quad (4)$$

3. POSITRONIUM EFFECTIVE ACTION

The quantization of fermion fields in action (4) yields generating functional:

$$\mathcal{Z} = \mathcal{N}_1 \int \mathbf{D}\bar{\psi} \mathbf{D}\psi e^{iS + i \int d^4x (\bar{\eta}\psi + \bar{\psi}\eta)}, \quad (5)$$

A_i here and below we consider only as *external fields*.

In this generating functional consider the term in (4)

$$\begin{aligned} & \frac{1}{2} \int d^4x d^4y j_\mu(x) D^{\mu\nu}(x-y) j_\nu(y) = \int d^4x \frac{1}{2} j_0 \frac{1}{\Delta} j_0 = \\ & = -\frac{1}{2} \int d^4x_1 d^4x_2 d^3\mathbf{y}_1 d^3\mathbf{y}_2 \bar{\psi}_{\alpha_1}(x_1) \psi^{\alpha_2}(x_2) \\ & \quad \underbrace{\gamma_{\alpha_2}^{0\alpha_1} \delta^4(x_1-x_2) \frac{e^2}{4\pi|\mathbf{x}_1-\mathbf{y}_2|} \delta^3(\mathbf{y}_1-\mathbf{y}_2) \gamma_{\beta_1}^{0\beta_2}}_{\mathcal{K}_{\beta_1\alpha_2}^{\alpha_1\beta_2}(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2)} \\ & \quad \bar{\psi}_{\beta_2}(x_2^0, \mathbf{y}_2) \psi^{\beta_1}(x_1^0, \mathbf{y}_1) = \\ & = -\frac{1}{2} \bar{\psi}_{a_1} \psi^{a_2} \mathcal{K}_{b_1 a_2}^{a_1 b_2} \bar{\psi}_{b_2} \psi^{b_1} = -\frac{1}{2} \bar{\psi}_{a_1} \psi^{b_1} \mathcal{K}_{b_1 a_2}^{a_1 b_2} \psi^{a_2} \bar{\psi}_{b_2} \end{aligned}$$

where in the first line $D^{\mu\nu}(x-y) = -\delta_0^\mu \delta_0^\nu \frac{e^2 \delta(x^0-y^0)}{4\pi|\mathbf{x}_1-\mathbf{y}_2|}$, and in the last line we use next notation: sum over pair indexes a_1, b_1 is summing over α_1, β_1 and integrating over $d^4x_1 d^3\mathbf{y}_1$.

So, there is forth power of ψ in generating functional. Hubbard-Stratonovich transformation allows to reduce power of ψ to second.

Let's consider the combination $\psi\bar{\psi}$ as a real bilocal field:

$$\chi_\beta^\alpha(x^0, \mathbf{x}, \mathbf{y}) = \psi^\alpha(x^0, \mathbf{x}) \bar{\psi}_\beta(x^0, \mathbf{y})$$

Transposition operation we define as:

$$\chi_\alpha^T{}^\beta(x^0, \mathbf{x}, \mathbf{y}) = -\chi_\alpha^\beta(x^0, \mathbf{y}, \mathbf{x})$$

Then introduce new bilocal field $\mathcal{M}^\alpha_\beta(x^0, \mathbf{x}, \mathbf{y})$ and make

$$\begin{aligned} e^{i \int d^4x \frac{1}{2} j_0 \frac{1}{\Delta} j_0} &= e^{-\frac{i}{2} \bar{\psi}_{a_1} \psi^{b_1} \mathcal{K}^{a_1}_{b_1 a_2}{}^{b_2} \psi^{a_2} \bar{\psi}_{b_2}} = \\ &= \mathcal{N} \int \mathcal{D}\mathcal{M} e^{\frac{i}{2} \mathcal{M}^T_{a_1}{}^{b_1} \mathcal{K}^{-1 a_1}_{b_1 a_2}{}^{b_2} \mathcal{M}^{a_2}_{b_2} + i \bar{\psi}_a \psi^b \mathcal{M}^a_b} \end{aligned}$$

$\sqrt{\det \mathcal{K}^{-1}}$ is constant and has been included in \mathcal{N} .

After that generating functional (5) takes form

$$\begin{aligned} \mathcal{Z} = \mathcal{N}_2 \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\mathcal{M} \exp i \left[\int d^4x \left(\frac{1}{2} (\dot{A}_i \dot{A}_i - B_i B_i) + A_i j_i - \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi \right) + \right. \\ \left. + \frac{1}{2} \mathcal{M}^T_{a_1}{}^{b_1} \mathcal{K}^{-1 a_1}_{b_1 a_2}{}^{b_2} \mathcal{M}^{a_2}_{b_2} + \bar{\psi}_a \psi^b \mathcal{M}^a_b + \int d^4x (\bar{\eta} \psi + \bar{\psi} \eta) \right] \end{aligned}$$

Introduce notation:

$$G_{mA}^{-1 \alpha}_{\beta}(x, y) \equiv (-i \gamma^{\mu \alpha}_{\beta} \partial_\mu + m \delta^\alpha_{\beta} + e A_i \gamma^{i \alpha}_{\beta}) \delta^4(x - y) + \mathcal{M}^\alpha_\beta(x^0, \mathbf{x}, \mathbf{y}) \delta(x^0 - y^0)$$

and define inverse operator as:

$$\int d^4y G_{mA}^{-1 \alpha}_{\beta}(x, y) G_{mA}^{\beta}_{\gamma}(y, z) = \delta^\alpha_{\gamma} \delta^4(x - z)$$

that allows us to write generating functional in more compact form:

$$\begin{aligned} \mathcal{Z} = \mathcal{N}_2 \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\mathcal{M} \exp i \left[\int d^4x \left(\frac{1}{2} (\dot{A}_i \dot{A}_i - B_i B_i) \right) + \frac{1}{2} \mathcal{M}^T_{a_1}{}^{b_1} \mathcal{K}^{-1 a_1}_{b_1 a_2}{}^{b_2} \mathcal{M}^{a_2}_{b_2} + \right. \\ \left. + \int d^4x d^4y \bar{\psi}_\alpha(x) G_{mA}^{-1 \alpha}_{\beta}(x, y) \psi^\beta(y) + \int d^4x (\bar{\eta} \psi + \bar{\psi} \eta) \right] \end{aligned}$$

After integration over fermions finally we have:

$$\begin{aligned} \mathcal{Z} = \mathcal{N}_3 \int \mathcal{D}\mathcal{M} \exp i \left[\int d^4x \left(\frac{1}{2} (\dot{A}_i \dot{A}_i - B_i B_i) \right) + \frac{1}{2} \mathcal{M}^T_{a_1}{}^{b_1} \mathcal{K}^{-1 a_1}_{b_1 a_2}{}^{b_2} \mathcal{M}^{a_2}_{b_2} - \right. \\ \left. - \int d^4x d^4y \bar{\eta}_\alpha(x) G_{mA}^{\alpha}_{\beta}(x, y) \eta^\beta(y) - i \text{tr} \ln G_{mA}^{-1} \right] \end{aligned}$$

So, effective action for positronium field take form:

$$S_P = \int d^4x \left(\frac{1}{2} (\dot{A}_i \dot{A}_i - B_i B_i) \right) + \frac{1}{2} \mathcal{M}^T_{a_1}{}^{b_1} \mathcal{K}^{-1 a_1}_{b_1 a_2}{}^{b_2} \mathcal{M}^{a_2}_{b_2} - i \text{tr} \ln G_{mA}^{-1} \quad (6)$$

4. QUANTIZATION OF BILOCAL FIELDS

For our purpose it would be enough to use the semi-classic approach. According to this approach the first variation of the action (6) is the Schwinger-Dyson equation :

$$\left. \frac{\delta S_P}{\delta \mathcal{M}} \right|_{A=0} = 0 \quad (7)$$

that gives a fermion spectrum. And the second variation in the point of minimum is the Bethe-Salpeter equation:

$$\frac{\delta^2 S_P}{\delta \mathcal{M}^2} \Gamma = 0$$

which allows us to find a bound state spectrum.

4.1. Schwinger-Dyson equation

Denote solution of (7) as $\Sigma^\alpha_\beta(x^0, \mathbf{x}, \mathbf{y}) - m\delta^\alpha_\beta \delta(\mathbf{x} - \mathbf{y})$. Than introduce notation:

$$G_\Sigma^{-1\alpha}_\beta(x, y) \equiv -i\gamma^{\mu\alpha}_\beta \partial_\mu + \Sigma^\alpha_\beta(x^0, \mathbf{x}, \mathbf{y}) \delta(x^0 - y^0),$$

where the inverse operator is defined as

$$\int d^4y G_\Sigma^{-1\alpha}_\beta(x, y) G_\Sigma^\beta_\gamma(y, z) = \delta^\alpha_\gamma \delta^4(x - z).$$

The equation (7) now takes the form:

$$\Sigma^{a_1}_{b_1} = m\delta^{a_1}_{b_1} + i\mathcal{K}^{a_1}_{b_1 a_2} G_\Sigma^{a_2}_{b_2}, \quad (8)$$

in the last term the sum is over pair indexes a_2, b_2 is summing over α_2, β_2 and integrating over $d^4x_2 d^4y_2 \delta(x_2^0 - y_2^0)$. Below we also assume a delta-function $\delta(x_2^0 - y_2^0)$ in contraction of \mathcal{K} and G .

After Fourier transform let us takes an ansatz:

$$\Sigma(p, q) = \delta^4(p - q) \left(\hat{\mathbf{q}} + E(\mathbf{q}) e^{-2\frac{\hat{\mathbf{q}}}{|\mathbf{q}|} \vartheta(\mathbf{q})} \right).$$

Then we can inverse G_Σ^{-1} :

$$G_\Sigma = \left(\frac{\Lambda_+(\mathbf{q})}{q_0 - E(\mathbf{q}) + i\varepsilon} + \frac{\Lambda_-(\mathbf{q})}{q_0 + E(\mathbf{q}) - i\varepsilon} \right) \gamma^0, \quad (9)$$

where:

$$\Lambda_\pm(\mathbf{q}) = e^{\frac{\hat{\mathbf{q}}}{|\mathbf{q}|} \vartheta(\mathbf{q})} \frac{1 \pm \gamma^0}{2} e^{-\frac{\hat{\mathbf{q}}}{|\mathbf{q}|} \vartheta(\mathbf{q})}. \quad (10)$$

Well known that in case of QED Schwinger-Dyson equation gives small correction to electron mass, so we take the solution of Schwinger-Dyson equation in form $\Sigma_b^a \approx m\delta_b^a$, where m is the electron mass.

4.2. Bethe-Salpeter equation

Taking the second variation of the action (6) in the point of minimum that we have found solving the Schwinger-Dyson equation (8):

$$\left. \frac{\delta^2 S_P}{\delta \mathcal{M}^2} \right|_{A=0, \mathcal{M}=\Sigma-m} \Gamma = 0$$

we obtain the Bethe-Salpeter equation:

$$\Gamma_{b_1}^{a_1} = i\mathcal{K}_{b_1 a_2}^{a_1 b_2} G_{\Sigma}^{a_2 a_3} \Gamma_{b_3}^{a_3} G_{\Sigma}^{b_3 b_2}.$$

After Fourier transform and inserting G_{Σ} in (9), let us take integral over q_0 :

$$\Gamma(\mathbf{p}) = \int d^3\mathbf{q} \mathcal{K}(\mathbf{p}-\mathbf{q}) \Psi(\mathbf{q}) \quad (11)$$

where the wave-function:

$$\Psi(\mathbf{q}) = \gamma^0 \left(\frac{\bar{\Lambda}_+(\mathbf{q}) \Gamma(\mathbf{q}) \Lambda_-(\mathbf{q})}{E_P - m_P + i\varepsilon} + \frac{\bar{\Lambda}_-(\mathbf{q}) \Gamma(\mathbf{q}) \Lambda_+(\mathbf{q})}{E_P + m_P - i\varepsilon} \right) \gamma^0 \quad (12)$$

where analogously to (10) one has:

$$\bar{\Lambda}_{\pm}(\mathbf{q}) = e^{-\frac{\hat{\mathbf{q}}}{|\mathbf{q}|} \vartheta(\mathbf{q})} \frac{1 \pm \gamma^0}{2} e^{\frac{\hat{\mathbf{q}}}{|\mathbf{q}|} \vartheta(\mathbf{q})}. \quad (13)$$

Acting (13) and (10) on (12), and inserting (11) in (12) we have equation for Ψ :

$$(E_P \mp m_P) \Lambda_{\pm}(\mathbf{p}) \Psi(\mathbf{p}) \bar{\Lambda}_{\mp}(\mathbf{p}) = \Lambda_{\pm}(\mathbf{p}) \left(\int d^3\mathbf{q} \mathcal{K}(\mathbf{p}-\mathbf{q}) \Psi(\mathbf{q}) \right) \bar{\Lambda}_{\mp}(\mathbf{p})$$

If $|\mathbf{p}| \ll m$ than this equation turns into the Schrodinger one [12]:

$$\left(\frac{1}{2m} \mathbf{p}^2 + (2m - m_P) \right) \Psi(\mathbf{p}) = \int d^3\mathbf{q} \mathcal{K}(\mathbf{p}-\mathbf{q}) \Psi(\mathbf{q}) \quad (14)$$

As we interested in parapositronium, we can take the next ansatz:

$$\Psi(\mathbf{q}) = N \gamma^5 \gamma^0 \psi(\mathbf{p}) \quad (15)$$

where N is unknown normalized factor and will be defined below, $\psi(\mathbf{p})$ is the nonrelativistic wave-function – the solution of Schrodinger equation (14):

$$\psi(r) = \frac{1}{\sqrt{\pi}} \left(\frac{m\alpha}{2} \right)^{\frac{3}{2}} e^{-r\frac{m\alpha}{2}} \quad \rightarrow \quad \psi(\mathbf{p}) = \frac{1}{\pi} \frac{(m\alpha)^{\frac{5}{2}}}{\left(\left(\frac{m\alpha}{2} \right)^2 + \mathbf{p}^2 \right)^2} \quad (16)$$

Because of smallness α the condition $|\mathbf{p}| \ll m$ is satisfied. Therefore the motion of an electron and a positron in the positronium is nonrelativistic and we can take $M_P \simeq 2m$.

5. TRIANGLE ANOMALY

Small variation action S_P (6) over \mathcal{M} and double variation over A give us

$$\mathcal{M}' A_m A_k \frac{\delta}{\delta \mathcal{M}} \frac{\delta}{\delta A_m} \frac{\delta}{\delta A_k} \Big|_{A=0, \mathcal{M}=\Sigma-m} S_P = -ie^2 A_m A_k \text{tr} \left(\gamma^m G_\Sigma \gamma^k G_\Sigma \mathcal{M}' G_\Sigma \right),$$

that can be interpreted as the triangle anomaly (fig.1) with an amplitude

$$M_\Delta = e^2 \int d^4 q \text{tr} \left[G_\Sigma \left(q + \frac{\mathcal{P}}{2} \right) \Gamma(q) G_\Sigma \left(q - \frac{\mathcal{P}}{2} \right) \hat{A}_1(k_1) G_\Sigma \left(\frac{k_2 - k_1}{2} + q \right) \hat{A}_2(k_2) \right]$$

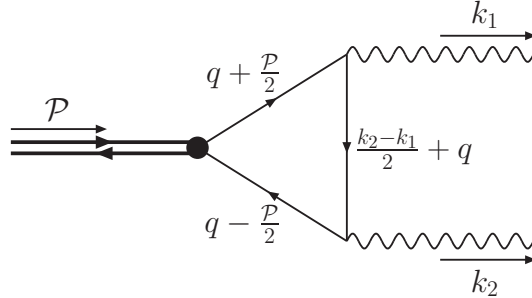


Figure 1. Triangle anomaly.

Below in our calculations we consider one virtual photon with momentum k_1 and other real photon with momentum k_2 . For simplicity the reference frame is chosen as a positronium rest frame $\mathcal{P} = (M_P, 0, 0, 0)$. Let us denote \mathbf{k}_2 as \mathbf{k} .

Rather crude estimate can be done in the following way. Using (9), let us integrate over q_0 and take into account only a pole that forms the wave-function (12), using (15)

$$M_\Delta = e^2 N \int d^3 \mathbf{q} \psi(\mathbf{q}) \text{tr} \left(\gamma^5 \gamma^0 \hat{\mathbf{A}}_1(k_1) \frac{\frac{\hat{k}_2 - \hat{k}_1}{2} + \hat{q} + m}{2 \left(|\mathbf{k}| \sqrt{m^2 + \mathbf{q}^2} - \mathbf{k} \mathbf{q} \right)} \hat{\mathbf{A}}_2(k_2) \right).$$

Let us make a change γ^0 for $\frac{\hat{P}^0}{M_P}$ and make the trace over the gamma matrixes product

$$M_{\Delta} = -4e^2 N \int d^3\mathbf{q} \psi(\mathbf{q}) \varepsilon_{0\lambda\mu\nu} e_1^{\mu} e_2^{\nu} \frac{\frac{k_2^{\lambda} - k_1^{\lambda}}{2} + q^{\lambda}}{2 \left(|\mathbf{k}| \sqrt{m^2 + \mathbf{q}^2} - \mathbf{k}\mathbf{q} \right)} \frac{\mathcal{P}^0}{M_P}.$$

The term with q^{λ} gives zero contribution. We can change $\varepsilon_{0\lambda\mu\nu} \frac{\mathcal{P}^0}{M_P}$ for sum $\varepsilon_{\rho\lambda\mu\nu} \frac{\mathcal{P}^{\rho}}{M_P}$, furthermore we can suppose now that in arbitrary frame of reference (not in positronium rest frame of reference as we do before) the last sum should appear. Using (16) one gets:

$$M_{\Delta} = 4e^2 N \varepsilon_{\mu\nu\lambda\rho} k_1^{\mu} e_1^{\nu} k_2^{\lambda} e_2^{\rho} \frac{1}{2M_P} \frac{(m\alpha)^{\frac{5}{2}}}{\pi} \int d^3\mathbf{q} \frac{1}{\left(\left(\frac{m\alpha}{2} \right)^2 + \mathbf{q}^2 \right)^2} \frac{1}{\left(|\mathbf{k}| \sqrt{m^2 + \mathbf{q}^2} - \mathbf{k}\mathbf{q} \right)}$$

Notice, that as α is small, so only values of \mathbf{q} near zero are important. Therefore, we can write in spherical variables, where $\zeta = -\cos\theta$,

$$\begin{aligned} 2\pi \int_0^{\infty} dq \int_{-1}^1 dx \frac{q^2}{\left(\left(\frac{m\alpha}{2} \right)^2 + q^2 \right)^2} \frac{1}{\left(k\sqrt{m^2 + q^2} - kq\zeta \right)} &\simeq \\ &\simeq 2\pi \int_{-1}^1 dx \frac{1}{\left(k\sqrt{m^2 + q^2} - kq\zeta \right)} \Bigg|_{q \rightarrow 0} \int_0^{\infty} dq \frac{q^2}{\left(\left(\frac{m\alpha}{2} \right)^2 + q^2 \right)^2} = 2\pi \frac{2}{km} \frac{\pi}{2\alpha m}, \end{aligned}$$

and

$$M_{\Delta} = \frac{2N\alpha^{\frac{5}{2}}}{\sqrt{m}k} \varepsilon_{\mu\nu\lambda\rho} k_1^{\mu} e_1^{\nu} k_2^{\lambda} e_2^{\rho}.$$

To get agreement with the decay $P \rightarrow \gamma\gamma$ given by anomaly, we should take the normalization factor as

$$N = \sqrt{2\pi} \sqrt{\frac{m}{2}} \frac{1}{2}.$$

Thus finally we have for the triangle diagram

$$M_{\Delta} = C_P \varepsilon_{\mu\nu\lambda\rho} k_1^{\mu} e_1^{\nu} k_2^{\lambda} e_2^{\rho}, \quad (17)$$

where

$$C_P = \frac{4m\sqrt{\pi}\alpha^{\frac{5}{2}}}{4m^2 - k_1^2} = \frac{2m\sqrt{\pi}\alpha^{\frac{5}{2}}}{(k_2\mathcal{P})}. \quad (18)$$

6. HAMILTONIAN ANOMALIES

In quantization of the Hamiltonian systems we need canonical coordinates and momenta. If the triangle anomaly exists, it changes the momentum. Consider as an example:

$$S_{example} = \int d^4x \left(\frac{1}{2}(\dot{A}_i \dot{A}_i - B_i B_i) + C \phi \dot{A}_i B_i \right)$$

which is the action (6) only with the first term and the triangle anomaly (17) with $\mathcal{P} = (M_P, 0, 0, 0)$ and $C_P = C$ – consider as a constant, ϕ is the wave function of positronium.

The canonicals momentum:

$$E_i = \frac{\delta S_P}{\delta \dot{A}_i} \neq \dot{A}_i,$$

so, one can write:

$$S_{example} = \int d^4x \left(\frac{(\dot{A}_i + C \phi B_i)^2 - B_i B_i}{2} - \frac{1}{2} C^2 \phi^2 B_i B_i \right)$$

and consider the first term as a "renormalized" electromagnetic field and the second term as a Hamiltonian anomaly shown in fig.2.

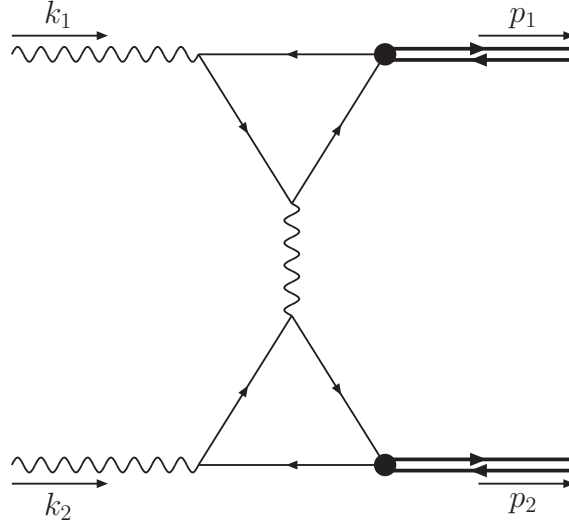


Figure 2. Hamiltonian anomaly.

Back to the action (6) and triangle anomaly (17,18), the amplitude of the Hamiltonian anomaly (fig.2) is

$$M_{\times} = -\frac{1}{2} \frac{C_P(p_1, k_1) C_P(p_2, k_2)}{M_P^2} \varepsilon_{\mu\nu\lambda\rho} p_1^{\mu} k_1^{\lambda} e_1^{\rho} \varepsilon^{\sigma\nu\xi\zeta} p_{2\sigma} k_{2\xi} e_{2\zeta}.$$

Evaluating sum over ν

$$M_{\times} = -\frac{1}{2} \frac{C_P(p_1, k_1) C_P(p_2, k_2)}{M_P^2} (V e_1^\rho e_{2\rho} + T^{\mu\nu} e_{1\mu} e_{2\nu}), \quad (19)$$

where

$$\begin{aligned} V &= -(p_1 p_2)(k_1 k_2) + (p_1 k_2)(k_1 p_2), \\ T^{\mu\nu} &= (p_1 p_2) k_2^\mu k_1^\nu - (k_1 p_2) k_2^\mu p_1^\nu - (p_1 k_2) p_2^\mu k_1^\nu + (k_1 k_2) p_2^\mu p_1^\nu. \end{aligned} \quad (20)$$

The square of the amplitude equals to

$$\begin{aligned} |M_{\times}|^2 &= \frac{\pi^2 \alpha^{10}}{4(k_1 p_1)^2 (k_2 p_2)^2} \left(2(p_1 p_2)(k_1 k_2)(p_1 k_1)(p_2 k_2) + 2(p_1 k_2)(k_1 p_2)(p_1 k_1)(p_2 k_2) - \right. \\ &\quad \left. - 2M_P^2(p_2 k_1)(p_2 k_2)(k_1 k_2) - 2M_P^2(k_1 k_2)(p_1 k_1)(p_1 k_2) + M_P^4(k_1 k_2)^2 \right). \end{aligned}$$

After an integration over the positronium momenta p_1 and p_2 , finally the cross-section is

$$\sigma = \frac{\pi \alpha^{10}}{96 M_P^2 s} \left((2s + 7M_P^2) \sqrt{1 - \left(\frac{2M_P}{s} \right)^2} - 6M_P^2 \ln \frac{\sqrt{s} + \sqrt{s - (2M_P)^2}}{\sqrt{s} - \sqrt{s - (2M_P)^2}} \right),$$

where $s = (k_1 + k_2)^2$ and M_P – positronium mass.

Consider the behavior of σ at small and large s in the center of mass frame of reference.

Denote the energy of one photon as E_γ , then $s = 4E_\gamma^2$. For $E_\gamma \rightarrow M_P$ we have:

$$\sigma = \frac{\pi \alpha^{10}}{128 M_P^2} \sqrt{1 - \left(\frac{M_P}{E_\gamma} \right)^2},$$

and for $E_\gamma \rightarrow \infty$ we have:

$$\sigma \rightarrow \frac{\pi \alpha^{10}}{48 M_P^2}.$$

In conclusion we should notice that in the process $\gamma\gamma \rightarrow PP$, where P is the parapositronium besides (19,20) there are leading contributions from the box diagram.

7. CONCLUSION

In this paper we obtain the bound state generation functional by Poincaré-invariant irreducible representations. Specific feature of this method is an absence of explicit relativistic covariance. An attempt to conserve the covariance based on Markov-Yukawa constraint was considered in [13]. We will prolong this research in future papers. Some comments about

the relativistic covariant excluding the time component discussed in section 2 can be found in the review [6].

The quantization of the positronium action (6) is made in the semi-classical approach.

As an example of this method, we calculated the triangle anomaly (fig.1). To simplify the calculations the assumption about poles was made. Besides in resulting the answer (17,18) was made an assumption about relativistic structure of the answer.

The anomaly triangle diagrams change canonical momenta in Hamiltonian systems. That may leads to additional contribution (19,20) to the process $\gamma\gamma \rightarrow PP$. We can calculate this contribution using the previous results with the triangle diagram.

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